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Optimal Nonlinear Feedback Control Design Using a Waypoint Method

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This paper discusses an innovative idea of blending the notion of a waypoint scheme with a series solution method developed by the authors for solving the Hamilton–Jacobi–Bellman equation in the context of designing optimal feedback control laws for nonlinear dynamic systems subject to terminal constraints. The overall time interval of the given problem is partitioned into smaller segments, and the series solution method is applied within each segment using stored gains that are computed from one segment only. The methodology is applied to highly nonlinear systems including a minimum orbit transfer problem. Several examples are demonstrated, and the results are compared with the corresponding open-loop solutions to demonstrate the efficacy of the proposed method.

Introduction

BELLMAN's dynamic programming approach [1,2] for traditional optimal control problems leads to the Hamilton–Jacobi–Bellman (HJB) equation. This is a partial differential equation whose solution is the optimal cost for a dynamical system that is controlled to a final condition from a starting point in the state space. Assuming the continuity and differentiability of the optimal cost function, the solution of HJB equation satisfies both the necessary and sufficient conditions for optimality. The solution depends on the instantaneous states and time, and it can be used to generate the controls. Therefore, one obtains optimal controls in feedback form using this approach. Computing the solution to an HJB equation can be challenging, however, because the solution may be discontinuous. Thus, computational methods for solving the HJB equation remain an active area of research.

Various methods to solve the HJB equation have been presented in the literature [3–8]. Some methods may be collectively called iterative schemes. This set mainly includes the Approximating Sequence of Riccati Equations method [9], Galerkin approximations techniques [10–12], level-set methods [13], finite volume methods [14], the generalized HJB equation method [15], and neural network methods [16]. Other methods may be collectively called noniterative schemes. This includes the State-Dependent Riccati Equation method [17], high-order series expansion methods [18,19], and generating-function canonical-transformations methods [20]. All of these methods have been demonstrated in special cases, notably infinite-time horizon problems and problems with linear terminal constraints. They are not suited, however, for problems with nonlinear terminal constraints. Conversely, a recently developed series solution method has been shown to be useful for problems subject to nonlinear terminal constraints. Vadali and Sharma [21] recently presented a series solution method (SSM) to synthesize nonlinear feedback solutions to finite time, nonlinear optimal control problems subject to nonlinear terminal constraints. The SSM can be considered a generalization of the well-known sweep method that is tailored to handle nonlinear polynomial systems with nonlinear terminal constraints. Broadly speaking, there are two aspects of the SSM that significantly influence the accuracy of the computed feedback solution. The first is the order of the series to synthesize the feedback control gains, and the second is the order of a series reversion [22] to determine the Lagrange multipliers associated with the nonlinear terminal constraints. Low-order series expansions are preferable in both cases because low-order expansions lead to fewer terms. But low-order series expansions may not be able to fully capture nonlinear characteristics, so the SSM may have trouble converging to a solution. One remedy in this case is to increase the order of the expansion. Another remedy is to carve the problem into a collection of better-suited subproblems. This is the strategy behind multiple-shooting methods for solving two-point boundary value problems [23], and a feedback form of this philosophy is pursued in this work. The waypoint method divides the given time domain into smaller, equal length time intervals. The optimal nonlinear feedback gains are computed and stored for one time interval and these are used in every time segment. Interior conditions that must be satisfied are always created when a problem is broken into subproblems, and things are no different here. In this case, the interior conditions are found from an optimum criterion.

The paper begins with an overview of the SSM. The waypoint method is then introduced and demonstrated on two example problems. Einstein's index notation is sometimes used in this paper. The summation convention is that if any index is repeated twice within a term, then the term represents the summation of every possible value of the index. An index must not be repeated more than twice in a term. An index that appears only once in a term is a free index, and the equation is valid for each possible value of the index.

Brief Overview of the Series Solution Method

The terminally constrained optimal control problem statement is given as follows:

$$\text{Minimize}: J = \int_{t_0}^{t_f} L(x, u, t)\,\mathrm{d}t \tag{1}$$

$$\text{subject to } \dot{x} = f(x, u, t); x(t_0) = x_0$$

$$\psi \equiv g(x(t_f), t_f) = \psi_f \tag{2}$$

where $x \in \Re^{n\times 1}$, $u \in \Re^{m\times 1}$ and $\psi \in \Re^{s\times 1}$ are vectors representing the states, affine controls, and convex terminal constraints at the final

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time t_f, respectively. The value of the constraint is defined as ψ_f, to satisfy Eq. (2). In the optimal control literature, such a terminal constraint is also known as a hard constraint, which can be specified as an explicit point constraint or a hyper-surface. Moreover $f(x, u, t)$ is a smooth, vector-valued analytic function and x_0 is the prescribed initial condition at the initial time t_0.

One common procedure to solve the optimal control problem described by Eqs. (1) and (2) begins by using the calculus of variations approach to derive the necessary conditions for optimality. The necessary conditions include the Euler–Lagrange equations and the transversality conditions [23,24], which taken together describe a two-point boundary value problem (TPBVP). The solution to the TPBVP can be found using direct or indirect methods [25]; both can produce an open-loop solution to the optimal control problem.

A different approach focuses on a feedback form, or closed-loop solution, to the optimal control problem. The feedback approach begins by forming the HJB equation, which may be compactly written as

$$\frac{\partial \Im^*}{\partial t} = -\min_u \left[L(x, u, t) + \left(\frac{\partial \Im^*}{\partial x} \right)^T f(x, u, t) \right] \tag{3}$$

with the boundary condition

$$\Im^*(x(t_f), t_f) = 0; x(t_f) \quad \text{such that } \psi(x(t_f)) = \psi_f \tag{4}$$

The goal here is to determine $\Im^*(x(t), t)$, which is commonly known as the optimal return function. The terminal constraints in the problem definition are associated with terminal Lagrange multipliers, ν, that are constant over time. The relationship between the costate vector $\lambda(t)$ and the gradient of the value function that connects dynamic programming and the calculus of variation approaches is:

$$\lambda(t) = \frac{\partial \Im^*(x(t), t)}{\partial x(t)} \tag{5}$$

A key relationship between the terminal Lagrange multipliers and optimal return function is established in [21]

$$\frac{\partial \Im^*(x(t), t)}{\partial \nu} = 0 = (\psi(x(t_f)) - \psi_f) \tag{6}$$

The SSM is a procedure to approximately solve the HJB equation. The method is thoroughly discussed in [21] and consequently only highlights are presented here. Especially, in the context of terminally constrained optimal control problems, the method begins by assuming a polynomial series representation for $\Im^*(x(t), t)$ in terms of the states and the terminal Lagrange multipliers.

$$\begin{aligned} \Im^*(x(t), t) &= S_{1_{ij}}(t) x_i x_j + S_{2_{ijk}}(t) x_i x_j x_k + \ldots + P_{1_{pj}}(t) \nu_l x_j \\ &+ P_{2_{pqj}}(t) \nu_p \nu_q x_j + P_{3_{pij}}(t) \nu_p x_i x_j + \ldots - (\psi_f)_p \nu_p \\ &+ V_{1_{pq}}(t) \nu_p \nu_q + V_{2_{pqr}}(t) \nu_p \nu_q \nu_r + \ldots \text{H.O.T} \\ & i, j, k, l \ldots\ldots \text{etc.} = 1, 2, 3, \ldots, n, \\ & p, q, r \ldots\ldots \text{etc} = 1, 2, 3, \ldots, s \leq n \end{aligned} \tag{7}$$

The range for the indices on the states is n whereas the range for the indices on the terminal Lagrange multipliers is s. The coefficients $S_{1_{ij}}$, for example, are unknown time-dependent gains. This form for the optimal return function is substituted into the HJB equation, and this leads to governing differential equations for the gains. Furthermore, Eq. (6) is used to eliminate ν as a part of the solution procedure. Once the time varying gains are known, they can be used to determine the control from the necessary optimality condition, which is proportional to the partial derivative of $\Im^*(x, t)$ with respect to the states

$$u^*(x, t) \propto \frac{\partial \Im^*(x, t)}{\partial x} \tag{8}$$

Waypoint Method

The central idea of the waypoint method is to implement the SSM within small time segments so that low-order series expansions and reversions are adequate. The original time domain of the optimal control problem is divided into smaller, interconnected optimal control problems. The waypoint method establishes the proper interior conditions that must be satisfied to preserve the optimality of the solution with respect to the original problem. The smaller time segments can be any length, but there is a significant benefit vis-à-vis the control gains to divide the original time domain into equitemporal segments. The waypoint method is introduced via a theoretical example with one waypoint. Without loss of generality, the scheme can be applied for several waypoints.

A single waypoint at time t_1 is introduced by partitioning the original time domain, $[t_0, t_f]$ into two equal parts: $[t_0, t_1]$ and $[t_1, t_f]$. Thus, two terminally constrained optimal control problems are generated, each fitting into the framework described by Eqs. (1) and (2). This statement is true, however, only if the state variable constraint at the waypoint is specified such that it lays on the optimal trajectory resulting from the original problem. In general, this constraint value has to be determined to satisfy some optimality criteria fitting to the problem. The intermediate constraint at the waypoint is specified as follows:

$$\psi(t_1) \equiv x(t_1) - x_1 = 0 \tag{9}$$

where x_1 is an unknown that must be determined. In general, it should be noted that if the waypoint location is considered arbitrarily, the gains must be stored for the largest time segment to use it in all other segments. As such, choosing the waypoint location at the midpoint is not a requirement, but it allows for a significant reduction in the storage of gains, especially for autonomous dynamical systems. This important fact can be understood from the formulation of gain differential equations for autonomous systems. Here it is explained by integrating only one gain equation, the Riccati equation, formulated for a finite time optimal control problem (OCP) with a dynamic system, $\dot{x}_i = f(x_i, u) \equiv A_{1_{ij}} x_j + A_{2_{ijk}} x_j x_k + \ldots + Bu$; $\psi \equiv x(t_f) - x_f = 0$ and $L(x, u, t) = \frac{1}{2}(x^T Q x + u^T R u)$; $R > 0$, $Q \geq 0$.

In terms of a gain matrix, S, the Riccati equation can be given as

$$\dot{S} = -(SA_1 + A_1^T S - SBR^{-1}B^T S + Q) \tag{10}$$

If specific conditions are met, viz., A_1 is associated with linear term in the system and it is not an explicit function of time and the B, Q and R matrices are constant valued, then Eq. (10) can be written as

$$\mathrm{d}t = -\mathrm{d}S / (SA_1 + A_1^T S - SBR^{-1}B^T S + Q) \tag{11}$$

Now assuming there is only one waypoint, which is located at the midpoint t_1 of the given time interval, the integration of the gain with respect to time in each partitioned interval can be written as

$$\begin{aligned} &\int_{t_1}^{t} \mathrm{d}t = \int_{S(t_1)}^{S(t)} [-1/(SA_1 + A_1^T S - SBR^{-1}B^T S + Q)]\,\mathrm{d}S \\ &\quad \forall\ t \in [t_0, t_1] \\ &\int_{t_f}^{t} \mathrm{d}t = \int_{S(t_f)}^{S(t)} [-1/(SA_1 + A_1^T S - SBR^{-1}B^T S + Q)]\,\mathrm{d}S \\ &\quad \forall\ t \in [t_1, t_f] \end{aligned} \tag{12}$$

Because the end of each interval is treated as a point terminal constraint like the one mentioned in the original optimal control problem, the gain boundary conditions, which are derived via the transversality conditions [24], will be the same: $S(t_1) = S(t_f)$. Therefore, there is no need to compute the gains for each partitioned optimal control problem because the respective gains are identical functions of time.

The above discussion focused on problems with point wise or linear terminal constraints. The situation slightly changes for problems with nonlinear terminal constraints. Specifically, for an autonomous systems with nonlinear terminal constraints, the backward integration of the gain differential equations must be stored for the two time segments with boundary conditions derived from the following: 1) A linear interior constraint that appears at every waypoint, and 2) the boundary specified as a nonlinear terminal (hyper-surface) that appears in the final time segment.

In the solution procedure with N waypoints, to compute the value of the waypoints and further generate the feedback control, the first N time segments of the OCP would use the computed gains in the Nth time segment whereas the last segment of the OCP that includes the nonlinear terminal constraint will use the stored gains for the $(N+1)$ th time segment.

Waypoint Computation

A necessary condition for determining the optimal waypoint constraint value is presented in this section. If this constraint is chosen arbitrarily, then the total cost of following the trajectory from the initial point to the terminal point, via the selected waypoint, will be higher than the optimal cost. The least cost is obtained if the chosen waypoint lies on the optimal trajectory.

As discussed in the previous sections, because the SSM obtains differential equations for the gain elements by collecting the coefficients of like powered terms involving x and ν, an advantage of the SSM is that the state boundary conditions do not affect the gain differential equations. The optimal return function can be expressed as a function of states, terminal Lagrange multipliers and time-dependent gains

$$\Im^* = \Im^*(x, \nu, \text{gains}(t)) \tag{13}$$

In terms of the value of a waypoint, the detailed expression of Eq. (13) can be given as

$$\Im^* = \Im^*(x, x_1, \nu(x, x_1), \text{gains}(t)) \tag{14}$$

Because x_1 is the value of an optimal waypoint at time $t = t_1$, it will minimize the total cost of the trajectory via a necessary condition

$$\frac{\mathrm{d}\Im^*}{\mathrm{d}x_1} = 0 \tag{15}$$

This necessary condition results in a parameter optimization problem that can be solved in an outer loop. The generalized outer loop optimization problem for N number of waypoints can be stated as

$$\begin{aligned} &\text{Minimize: } \Im^*(x_0, x_i, \nu(x_0, x_i), \text{gains}(t)) \\ &\quad \text{where, the gains are known} \quad \forall\ t \in \{t_0, t_i\} \\ &i = 1 \ldots N, \qquad N \equiv \text{the total number of waypoints} \end{aligned} \tag{16}$$

In the following section, additional necessary conditions are presented. For simplicity, these conditions are presented in the context for one waypoint, but the method generalizes to many waypoints.

Derivation of Necessary Conditions

Figure 1 shows a family of trajectories with respect to time. The primary curve represents the trajectory for the optimal state waypoint value; whereas the secondary curves are represent feasible trajectories for nonoptimal waypoint choices. Let the waypoint be located at time A and let O and B denote the given initial and final conditions at t_0 and t_f, respectively.

The objective of the outer loop problem is to minimize the total cost $\Im^*_{OB}$ with respect to $x_1(t_1)$. The total cost $\Im^*_{OB}$ can be written as the sum of two costs

$$\Im^*_{OB} = \Im^*_{OA} + \Im^*_{AB} \tag{17}$$

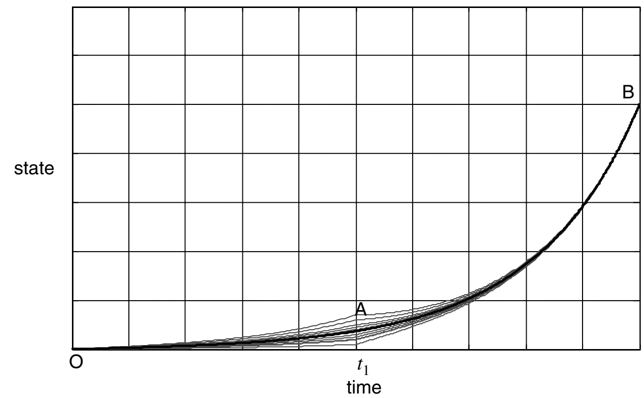

Fig. 1 Illustration of one waypoint along the trajectory.

Applying the necessary condition given by Eq. (15) yields

$$\frac{\mathrm{d}\Im^*_{OB}}{\mathrm{d}x_1} = \frac{\mathrm{d}\Im^*_{OA}}{\mathrm{d}x_1} + \frac{\mathrm{d}\Im^*_{AB}}{\mathrm{d}x_1} = 0 \tag{18}$$

$$\Rightarrow \frac{\partial\Im^*_{OA}}{\partial x_1} + \frac{\partial\Im^*_{OA}}{\partial \nu_O}\frac{\partial \nu_O}{\partial x_1} + \frac{\partial\Im^*_{AB}}{\partial x_1} + \frac{\partial\Im^*_{AB}}{\partial \nu_A}\frac{\partial \nu_A}{\partial x_1} = 0 \tag{19}$$

where $\nu_O = \nu(t_1^-)$ and $\nu_A = \nu(t_1^+)$ are terminal Lagrange multipliers in segments OA and AB, respectively. Substituting appropriate forms of Eq. (6) in Eq. (19), i.e., $\partial\Im^*_{OA}/\partial\nu_O = \partial\Im^*_{AB}/\partial\nu_A = 0$ results in the following:

$$\frac{\partial\Im^*_{OA}}{\partial x_1} + \frac{\partial\Im^*_{AB}}{\partial x_1} = 0 \tag{20}$$

Now, using Eq. (9) in the segment AB

$$\frac{\partial\Im^*_{OA}}{\partial x_1} + \frac{\partial\Im^*_{AB}}{\partial x(t_1^+)} = 0 \tag{21}$$

The assumption being made here is that the optimal return function $\Im^*$ is a smooth and continuous function of the state x.

From Eq. (7), it is known that $\Im^*_{OA}$ is a linear function of the waypoint constraint value, x_1 which implies that

$$\frac{\partial\Im^*_{OA}}{\partial x_1} = -\nu(t_1^-) \tag{22}$$

Then, continuing from Eq. (20)

$$\Rightarrow -\nu(t_1^-) + \frac{\partial\Im^*_{AB}}{\partial x(t_1^+)} = 0 \tag{23}$$

For the segment AB, Eq. (5) is

$$\lambda(t_1^+) = \frac{\partial\Im^*_{AB}}{\partial x(t_1^+)} \tag{24}$$

For the segment OA, the transversality condition [24] at $t = t_1^-$ can be stated as

$$\lambda(t_1^-) = \left[\frac{\partial\psi(x(t_1^-), t_1^-)}{\partial x(t_1^-)}\right]^T \nu(t_1^-) \tag{25}$$

Using Eq. (25) along with Eq. (9) gives

$$\lambda(t_1^-) = \nu(t_1^-) \tag{26}$$

Now, substituting Eqs. (24) and (26) in Eq. (23) results in the following:

$$\lambda(t_1^-) = \lambda(t_1^+) \tag{27}$$

The above analysis proves that the optimal waypoint state ensures the continuity of the costates and Lagrange multipliers at the segment boundary. For the set of arbitrary interior point constraints, the general necessary conditions are derived in [23] by using the calculus of variations. Interested readers can verify that the given derivation finds the same condition derived above as Eq. (27), if the interior point constraint is the optimal waypoint that satisfies the optimal solution of the TPBVP.

Complete Procedure with Waypoints

The waypoint method is a feedback equivalent of the multiple shooting method. Before proceeding to the numerical examples, this section explains the overall procedure to compute the feedback solution in the presence of waypoints. The full procedure can be stated as follows:

1) Partition the given time domain into smaller segments.

2) Apply the SSM to compute the gain differential equations.

3) Integrate backwards in the last time interval to store gains, if the terminal constraint is a point/linear terminal constraint and the time segments are of equal length.

4) In the outer loop, apply the static optimization procedure mentioned above to compute the values of the waypoints.

5) Once the waypoints are known, the closed-loop system can be simulated in conjunction with the higher-order feedback control law. In the first segment, the first waypoint acts like a terminal constraint; in the next subsequent segment, the previous waypoint is treated as an initial condition and the next waypoint becomes a terminal constraint.

Illustrative Numerical Examples

Two numerical examples are considered to demonstrate the usefulness of the waypoint based series solution method. The first example is to obtain the feedback control law by minimizing the energy of a 2-D nonlinear system subject to the terminal constraints.

Example 1: 2-D Nonlinear System

The OCP is given as

$$\text{Min}: J = \frac{1}{2}\int_0^3 (x^2 + \dot{x}^2 + u^2)\,\mathrm{d}t \tag{28}$$

$$\text{subject to } \ddot{x} = -1.8x^2 + 0.75x^3 + u;$$
$$x(0) = 0.1; \qquad \dot{x}(0) = 0.2; \tag{29}$$

$$\text{Terminal constraints}: \psi \in \Re^{2\times 1} \equiv x(3) = 0.5, \qquad \dot{x}(3) = 0.5 \tag{30}$$

Solution Procedure

For the given problem, the HJB equation is

$$-\frac{\partial \mathfrak{I}^*}{\partial t} = \min_u \left[\frac{1}{2}(x^2 + \dot{x}^2 + u^2) + \frac{\partial \mathfrak{I}^*}{\partial x}(\dot{x}) + \frac{\partial \mathfrak{I}^*}{\partial \dot{x}}(-1.8x^2 + 0.75x^3 + u)\right],$$
$$\mathfrak{I}^*(x(t_f), t_f) = 0 \tag{31}$$

By using the necessary conditions [23] for optimality, the optimal feedback control can be written as

$$u^* = -\frac{\partial \mathfrak{I}^*}{\partial \dot{x}} \tag{32}$$

Note that there is no linear term in x, in Eq. (29), which makes the OCP very complicated. Proceeding with the straightforward expansion of $\mathfrak{I}^*$ to sixth-order, it is found that the series reversion process, which computes the value of ν during the forward integration, causes a significant error in the computed control. Thus, the usual single-segment procedure is deemed unsuitable to handle the strong nonlinearity present in this system.

Next, the above OCP is solved by using two waypoints at $t_i = 1, 2$ s. As before, $\mathfrak{I}^*$ is assumed to be a polynomial series of sixth-order in the states and second order in ν. It contains a total of 115 gains. The gains are computed in the last time interval: $t = [2, 3]$. Additional necessary conditions are used (outer loop) to obtain the optimal waypoints at these time instants.

Numerical Results

The waypoint selection methodology in conjunction with the series solution results in the following:

$$x(1) = 0.14452004365, \qquad \dot{x}(1) = -0.030242702365 \tag{33}$$

$$x(2) = 0.17207973378, \qquad \dot{x}(2) = 0.1371940561405 \tag{34}$$

Upon forward integration with the closed-loop controller in place, the open-loop and feedback costs are obtained as shown below:

$$\mathfrak{I}^*_{OL} = 0.3266422, \qquad \mathfrak{I}^*_{FB} = 0.3269268 \tag{35}$$

To demonstrate the outer loop optimization process, several waypoints in the vicinity of the optimal are selected. Figure 2 shows $\mathfrak{I}^*$ at $t = 1$ and 2 s with respect to the arbitrary waypoints and the existence and locations of the optimal waypoints can be ascertained. Figure 3 presents several phase plots obtained by using arbitrary waypoints at $t_i = 1, 2$ s. The teal line shows the open-loop solution, which is associated with the optimal waypoints. The computed feedback solutions with arbitrary waypoints are shown in red. The cost for the trajectory with waypoints located on the open-loop trajectory is the least among the other trajectories considered.

Figure 4 shows the final solution obtained by using the optimal waypoints. The open-loop and feedback solutions obtained using two waypoints are nearly indistinguishable. The computed solution obtained without using waypoints is also shown in this figure and it shows considerable error in the midcourse. Figure 5 shows the cost-to-go comparison for the open-loop and feedback controls with and without waypoints.

Figure 6 captures the feedback nature of the waypoint methodology. In Fig. 6, the phase portraits are shown for the perturbed initial conditions within the range of $[-10, 10]\%$. The two waypoint feedback solutions are obtained by using the gains computed and stored in one segment for the nominal initial condition, $x(t_0) = [0.1, 0.2]$. To check the robustness of the method, the respective open-loop solutions are also plotted in the same figure. Additionally, to provide the comparative numerical details with respect to the perturbed initial conditions, Table 1 presents the computed solutions for the waypoint values from the outer loop

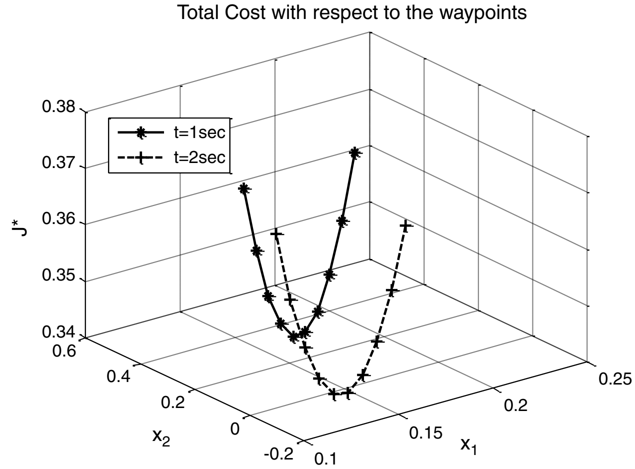

Fig. 2 Cost at $t = 1$ and 2 with arbitrary waypoints.

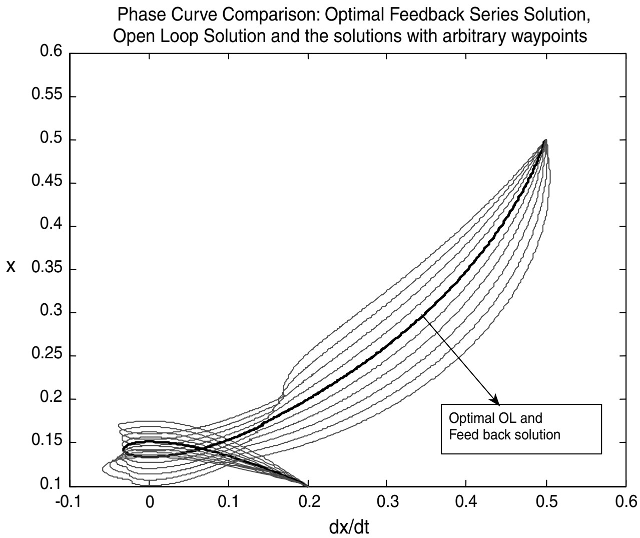

Fig. 3 Phase curve comparison with arbitrary waypoints at $t = 1$ and 2.

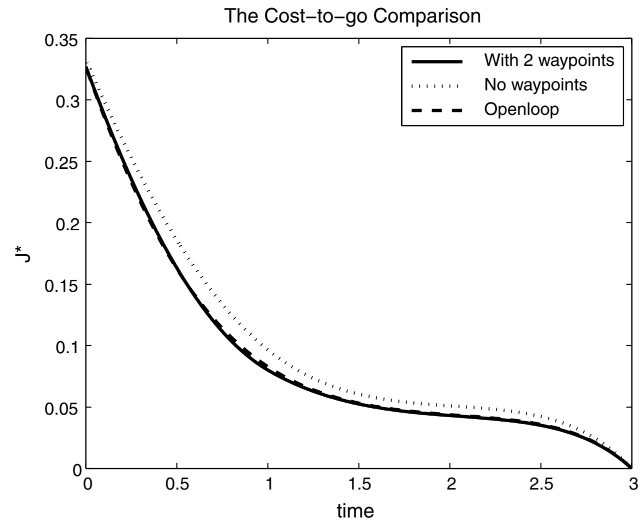

Fig. 5 Cost-to-go comparison with open-loop solution.

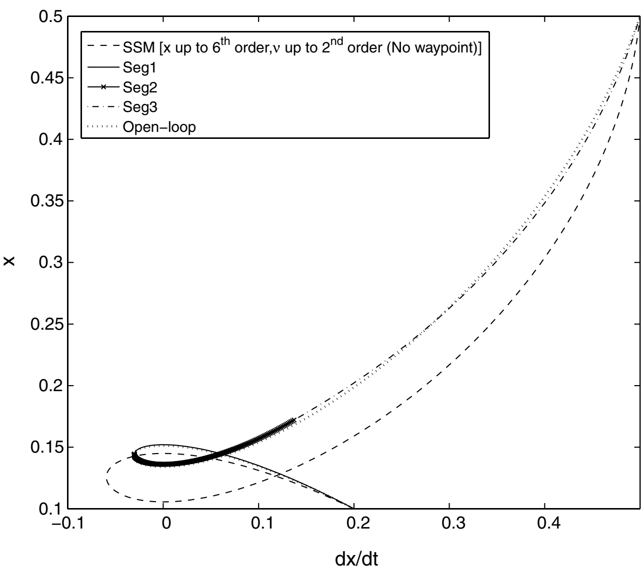

Fig. 4 Series solution with two waypoints.

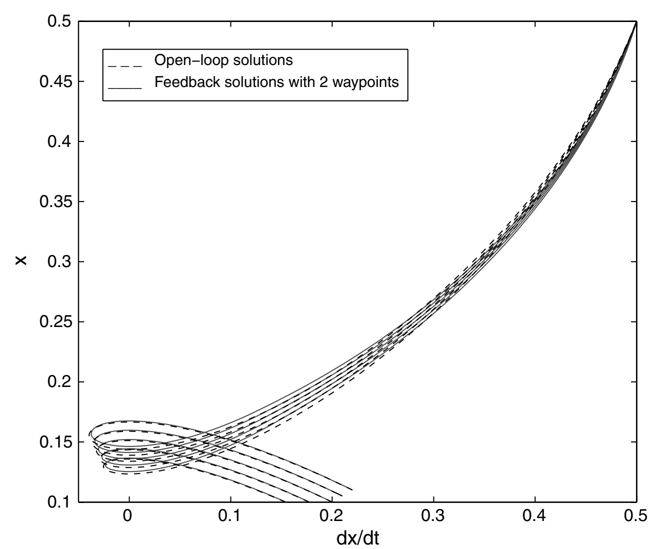

Fig. 6 Phase portraits with various initial conditions. Nominal initial condition is perturbed within the range of $[-10, 10]$%.

optimization and the cost-to-go along with the same obtained by using the open-loop method.

Next, the optimal control is slightly modified to include nonlinear terminal constraints. Specifically, the OCP given by Eqs. (28) and (29) is reconsidered with the following terminal constraints:

$$\psi \in \Re^{2\times 1} \equiv x(3) = 0.5, \qquad \dot{x}(3) - 0.1\dot{x}^2(3) = 0.475 \qquad (36)$$

A costate-based solution will converge to one of two solutions given by $\dot{x}(3) = 0.5$ or $\dot{x}(3) = 9.5$. These solutions exactly satisfy Eq. (36), thus the result obtained by using a traditional shooting method cannot be accepted as a local minimum until second-order conditions of optimality are tested.

The nonlinear terminal constraint in Eq. (36) makes the problem more challenging than before and in this case a successful use of the waypoint method required more waypoints as compared with the problem with linear or point terminal constraints. The feedback

Table 1 Waypoint values obtained in the outer loop compared with the solutions obtained by using open-loop method and the cost-to-go comparison for various initial conditions about nominal

Various initial conditions around $x(t_0) = [0.1, 0.2]$ with the stored gains		Optimal feedback solution for the first waypoint		Open-Loop solution for the first waypoint		Optimal feedback solution for the second waypoint		Open-Loop solution for the second waypoint		Cost-To-Go	
$x_1(t_0)$	$x_2(t_0)$	$x_1(t_1)$	$x_2(t_1)$	$x_1(t_1)$	$x_2(t_1)$	$x_1(t_2)$	$x_2(t_2)$	$x_1(t_2)$	$x_2(t_2)$	$\Im^*_{OL}$	$\Im^*_{FB}$
0.090	0.18	0.1306	−0.0235	0.1290	−0.0254	0.1673	0.1460	0.1664	0.1528	0.3129	0.3131
0.095	0.19	0.1376	−0.0269	0.1359	−0.0289	0.1697	0.1416	0.1688	0.1486	0.3196	0.3199
0.100	**0.20**	**0.1445**	**−0.0302**	**0.1428**	**−0.0324**	**0.1721**	**0.1372**	**0.1711**	**0.1443**	**0.3266**	**0.3269**
0.105	0.21	0.1515	−0.0336	0.1497	−0.0359	0.1745	0.1328	0.1735	0.1401	0.3339	0.3342
0.110	0.22	0.1585	−0.0370	0.1567	−0.0394	0.1769	0.1284	0.1758	0.1359	0.3415	0.3417

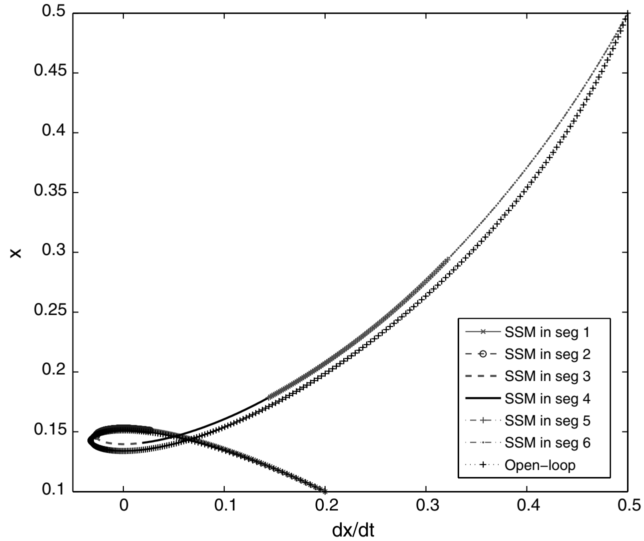

Fig. 7 An example with a nonlinear terminal constraint solved using SSM with five waypoints.

solution presented in Fig. 7 was developed by using the same $\Im^*$ expansion as used before for this OCP, but with five equally spaced waypoints. The interesting observation with this example is that the waypoint method not only computes the corrected higher-order feedback control that is based on using the SSM in small segments, but also exploits the advantages of the SSM as obtaining the trajectory with the lowest cost solution satisfying the terminal manifold accurately.

Example 2: Optimal Earth to Mars Orbit Transfer

The next example deals with a standard problem that has been considered by many researchers. The objective here is to validate the performance of the waypoint scheme on the problem of optimal orbit transfer from Earth to Mars in 144 days. The OCP can be stated as

$$\text{Minimize } J = \frac{1}{2}\int_0^{t_f}(u_2^2 + u_3^2)\,\mathrm{d}t \tag{37}$$

$$\text{subject to } \dot{r} = w; \qquad \dot{v}_T = -\frac{w v_T}{r} + u_2; \qquad \dot{w} = \frac{v_T^2}{r} - \frac{\mu}{r^2} + u_3 \tag{38}$$

$$r(0) = R = 1 \text{ AU}; \qquad v_T(0) = \sqrt{\frac{\mu}{R}}; \qquad w(0) = 0 \tag{39}$$

$$\begin{gathered}\text{Terminal constraints}: \psi \in \Re^{3\times 1} \equiv r(t_f) = R_f = 1.54 \text{ AU}; \\ v_T(t_f) = \sqrt{\frac{\mu}{R_f}}; \qquad w(t_f) = 0 \qquad t_f = 144 \text{ days}, \\ 1 \text{ AU} \equiv 1.4959965 \times 10^{11} \text{ m}; \qquad 1 \text{ TU} \equiv 58.132821 \text{ days}\end{gathered} \tag{40}$$

In the preceding equations, r, v_T, and w denote the radial position, the tangential velocity component, and the radial velocity component, respectively; u_2 is the control along the tangential direction and u_3 is the control along the radial direction. The constant μ is the gravitational parameter of the sun.

Solution Procedure

The Hamiltonian H can be written as

$$\begin{aligned} H = {} & \frac{1}{2}(u_2^2 + u_3^2) + \lambda_r(r) + \lambda_{v_T}\left(-\frac{v_T w}{r} + u_2\right) \\ & + \lambda_w\left(\frac{v_T^2}{r} - \frac{\mu}{r^2} + u_3\right) \end{aligned} \tag{41}$$

Since the SSM requires a polynomial form of the plant dynamics, the Hamiltonian of the system can be expanded about a reference orbit. In this solution, a circular orbit of the Earth radius, R is considered as a reference. Hence, H can be specified in terms of new coordinates defined below:

$$\delta x = \begin{bmatrix} r - R \\ v_T - \sqrt{\frac{\mu}{R}} \\ w - 0 \end{bmatrix} \tag{42}$$

The HJB equation for this example is given below:

$$\frac{\partial \Im^*}{\partial t} = -\min_{u_2,u_3}[H_1(\delta x, u_2, u_3, \lambda_r, \lambda_{v_T}, \lambda_w)] \tag{43}$$

where H_1 is a fourth-order Taylor series expansion of H with respect to δx.

By using the necessary condition, $\frac{\partial H_1}{\partial u_{2,3}} = 0$, the corresponding optimal controls are obtained as

$$u_2 = -\frac{\partial \Im^*}{\partial \delta x_2}; \qquad u_3 = -\frac{\partial \Im^*}{\partial \delta x_3} \tag{44}$$

Numerical Results and Discussion

Since the OCP is posed with hard terminal constraints, $\Im^*$ is expanded in a power series involving ν and δx. The regular series solution algorithm suffers from errors in the series reversion process due to the low order of the expansion about a circular orbit. Based upon the given transfer time, the computational difficulty of series reversion can be avoided by considering a larger expansion order of the cost-to-go for all the states but the expansion order for ν is limited to second order only. In such a case, the series reversion process is easy to implement but it results in a significant error in the midcourse

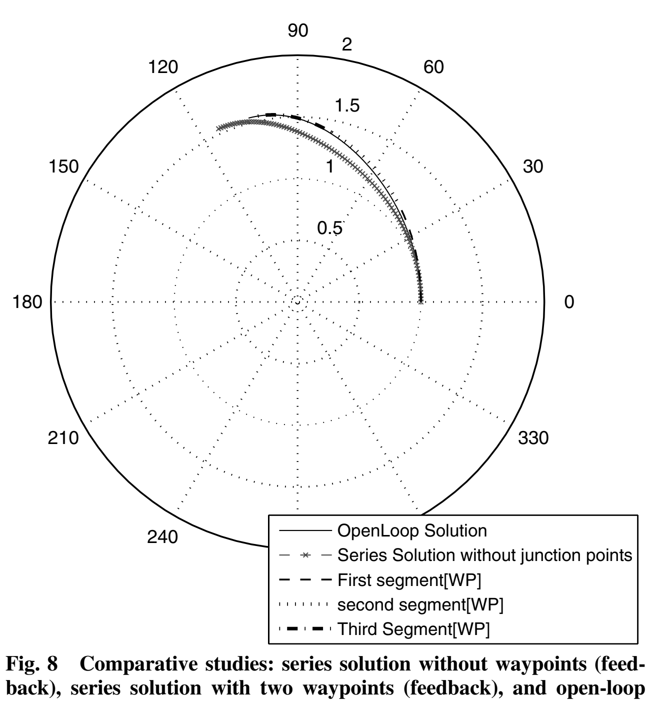

Fig. 8 Comparative studies: series solution without waypoints (feedback), series solution with two waypoints (feedback), and open-loop solution.

Table 2 The cost-to-go analysis for the orbit transfer problem

$\Im^*_{OL}$ (by using shooting method)	$\Im^*_{FB}$ (sixth order in x and second order in ν, no waypoint)	$\Im^*_{FB}$ (higher-order feedback with 2 waypoints)	$\Im^*_{FB}$ (higher-order feedback with 4 waypoints)	$\Im^*_{FB}$ (higher-order feedback with 12 waypoints)
0.09182	0.131412	0.10512	0.098204	0.094162

trajectory, which in turn leads to a higher cost-to-go. To bypass such problems with the assumed expansion of $\Im^*$, the waypoint scheme is employed to obtain more accurate results. The HJB equation is solved by using a sixth-order series expansion in x and second-order in ν, with 2, 4, and 12 equally spaced waypoints and the obtained solutions are compared with the open-loop solution. The following table shows the comparative study of $\Im^*$ for the different cases.

Figure 8 demonstrates polar plots of the transfer trajectory obtained by using three different methods: open-loop, series solution without waypoints, and series solution with two waypoints. Without the use of waypoints, the transfer follows a completely different midcourse trajectory from that of the open-loop solution. However, the inclusion of two waypoints provides a significant correction to the required control and midcourse trajectory as well.

Figures 9a and 9b show the transfer orbit with 4 and 12 waypoints. As the number of waypoints is increased from 4 to 12, the higher-order feedback solution depicted by the dotted bold line tends closer to the open-loop solution, shown by the dashed line. The cost-to-go analysis shown in Table 2 emphasizes that including more waypoints improves the midcourse solution. However, it should be noted that the choice of total number of waypoints depends upon the OCP and the nonlinearity of the dynamical system. The deviation in the midcourse trajectory can be reduced up to a certain extent by using a judicious combination of the waypoints and the number of terms allowed in the series expansion.

It is well known that ν becomes singular very close to the final time. Using many waypoints and short intervals exacerbates this singular behavior. It is found that considering 32 waypoints in the same orbit transfer problem increases the cost-to-go to 0.0966. Even if the computation of ν is stopped just before the terminal time, it causes an error in the waypoint calculation. Also, shrinking the time domain for each segment by increasing the number of waypoints does not allow the same gains to be used in each interval.

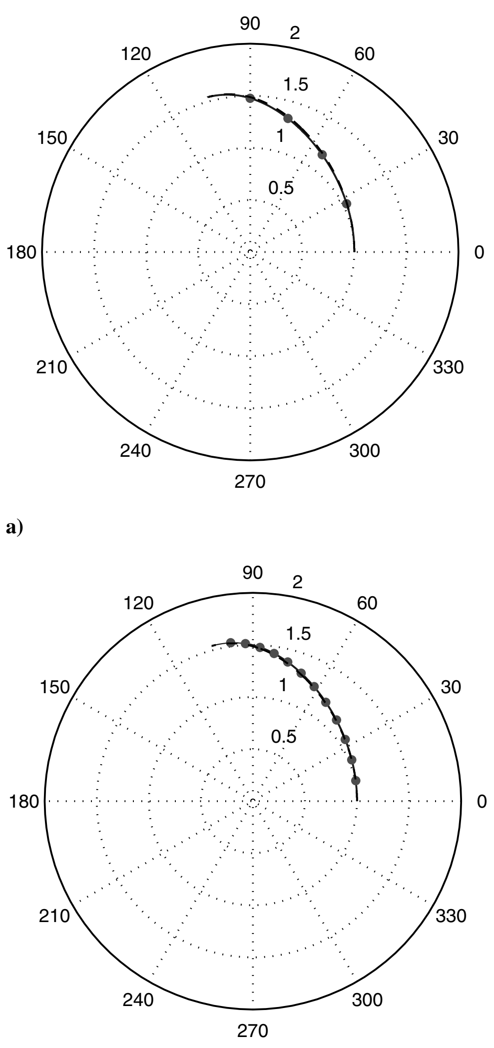

Fig. 9 Higher-order feedback solution with a) 4 waypoints, and b) 12 waypoints.

Conclusions

A waypoint method is presented to broaden the applicability of the SSM to finite time nonlinear terminally constrained problems. Computationally, the methodology is capable of reducing the gain storage requirements up to a certain extent. The novel algorithm is applied to a 2-D nonlinear example for which the regular SSM produces significant midcourse error. Numerical results from the waypoint scheme are validated by using the open-loop solutions and a thorough analysis is presented with respect to the perturbed initial conditions. An orbit transfer problem is solved by using multiple waypoints, and a cost analysis is presented to justify the scheme. It is demonstrated that highly accurate feedback solutions can be achieved with a proper combination of the waypoint method with a higher-order series solution. This methodology can be considered as the feedback equivalent to the multiple-shooting method.

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